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Cofinite modules and local cohomology [1]

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Abstract

We show that if M is a finitely generated module over a commutative Noetherian local ring R and I is a dimension one ideal of R (i.e., $\dim R/I = 1$), then the local cohomology modules $H_I^i(M)$ are I-cofinite; that is, $\mathrm{Ext}_R^j(R/I, H_I^i(M))$ is finitely generated for all i, j. We also show that if R is a complete local ring and P is a dimension one prime ideal of R, then the set of P-cofinite modules form an abelian subcategory of the category of all R-modules. Finally, we prove that if M is an n-dimensional finitely generated module over a Noetherian local ring R and I is any ideal of R, then $H_I^n(M)$ is I-cofinite. © 1997 Elsevier Science B.V.

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Let R be a commutative Noetherian local ring with maximal ideal m and let I be an ideal of R. An R-module N is said to be *I-cofinite* if $\mathrm{Supp}\, N \subseteq \mathrm{V}(I)$ and $\mathrm{Ext}_R^i(R/I, N)$ is finitely generated for all $i \geq 0$. Using Matlis duality one can show that a module is m-cofinite if and only if it is Artinian. As a consequence, the local cohomology modules $H_m^i(M)$ are m-cofinite for any finitely generated R-module M. In [6], Hartshorne posed the question of whether this statement still holds when m is replaced by an arbitrary ideal I; i.e., is $H_I^i(M)$ I-cofinite for all i? In general, the answer is no, even if R is a regular local ring. Let $R = k[[x, y, u, v]]$ be the formal power series ring in four variables over a field k, m the maximal ideal of R, $P = (x, u)R$ and $M = R/(xy - uv)$. Hartshorne showed that $\mathrm{Hom}_R(R/m, H_P^2(M))$ is not finitely generated, and hence $\mathrm{Hom}_R(R/P, H_P^2(M))$ cannot be finitely generated. In the positive direction, Hartshorne proved that if R is a complete regular local ring, P a dimension one prime ideal of

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R, and M a finitely generated R-module, then $H^i_P(M)$ is finitely generated for all i. In 1991, Huneke and Koh proved that if R is a complete local Gorenstein domain, I a dimension one ideal of R, and M a finitely generated R-module, then $H^i_I(M)$ is I-cofinite for all i [7, Theorem 4.1]. Recently, Delfino proved that the Gorenstein hypothesis in the Huneke–Koh theorem may be weakened to include all complete local domains R which satisfy one of the following conditions: (1) R contains a field; (2) if q is a uniformizing parameter for a coefficient ring for R then either $q \in \sqrt{I}$ or q is not in any prime minimal over I; or (3) R is Cohen–Macaulay [3, Theorem 3; 4, Theorem 2.21]. In this paper, we eliminate the complete domain hypothesis entirely by proving the following:

Theorem 1. *Let R be a Noetherian local ring, I a dimension one ideal of R, and M a finitely generated R-module. Then $H^i_I(M)$ is I-cofinite for all i.*

We prove this by establishing a change of ring principle for cofiniteness (Proposition 2) and then applying it to the Huneke–Koh result. Using this change of ring principle, we are also able to generalize Hartshorne's theorem that over a regular local ring, the P-cofinite modules (P a dimension one prime) form an abelian subcategory of the category of R-modules (Theorem 2).

We also prove a cofiniteness result about $H^n_I(M)$, where M is a finitely generated R-module and $n = \dim M$. In [12], Sharp proved that if R is a Noetherian local ring of dimension d and I is any ideal of R, then $H^d_I(R)$ is Artinian. From this it follows easily that if M is a finitely generated R-module of dimension n then $H^n_I(M)$ is Artinian (see also [10, Theorem 2.2]). Thus, $H^n_I(M)$ is m-cofinite. We prove that $H^n_I(M)$ is in fact I-cofinite (Theorem 3).

We begin the proof of Theorem 1 by proving the following generalization of [7, Lemma 4.2; 3, Lemma 2].

Proposition 1. *Let R be a Noetherian ring, M a finitely generated R-module and N an arbitrary R-module. Suppose that for some $p \geq 0$, $\operatorname{Ext}^i_R(M,N)$ is finitely generated for all $i \leq p$. Then for any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$, $\operatorname{Ext}^i_R(L,N)$ is finitely generated for all $i \leq p$.*

Proof. Using induction on p, we may assume that $\operatorname{Ext}^i_R(L,N)$ is finitely generated for all $i < p$ and all finitely generated modules L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$. (This is satisfied vacuously if $p = 0$.) By Gruson's Theorem [13, Theorem 4.1], given any finitely generated R-module L with $\operatorname{Supp} L \subseteq \operatorname{Supp} M$ there exists a finite filtration

$$0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

such that the factors L_i/L_{i-1} are homomorphic images of a direct sum of finitely many copies of M. By using short exact sequences and induction on n, it suffices to prove the case when $n = 1$. Thus, we have an exact sequence of the form

$$0 \to K \to M^n \to L \to 0$$

for some positive integer n and some finitely generated module K. This gives the long exact sequence

$$\cdots \to \mathrm{Ext}_R^{p-1}(K,N) \to \mathrm{Ext}_R^p(L,N) \to \mathrm{Ext}_R^p(M^n,N) \to \cdots$$

Since $\mathrm{Supp}\, K \subseteq \mathrm{Supp}\, M$ we have that $\mathrm{Ext}_R^{p-1}(K,N)$ is finitely generated (by the induction on p). As $\mathrm{Ext}_R^p(M^n,N) \cong \mathrm{Ext}_R^p(M,N)^n$ is finitely generated, the result follows. □

As a consequence, we have the following.

Corollary 1. *Let R be a Noetherian ring, I an ideal of R and N an R-module. The following are equivalent:*

(a) $\mathrm{Ext}_R^i(R/I,N)$ *is finitely generated for all* $i \geq 0$,

(b) $\mathrm{Ext}_R^i(R/J,N)$ *is finitely generated for all* $i \geq 0$ *and ideals* $J \supseteq I$,

(c) $\mathrm{Ext}_R^i(R/P,N)$ *is finitely generated for all* $i \geq 0$ *and all primes P minimal over I.*

Proof. We show that (c) implies (a). Let $P_1,\ldots,P_n$ be the minimal primes of I and $M = R/P_1 \oplus \cdots \oplus R/P_n$. Then $\mathrm{Ext}_R^i(M,N)$ is finitely generated for all i. As $\mathrm{Supp}\, R/I = \mathrm{Supp}\, M$, $\mathrm{Ext}_R^i(R/I,N)$ is finitely generated for all i by the proposition. □

The next result concerns spectral sequences, for which we use the notation from Ch. 5 of [14]. The essential idea for this lemma can be found in the proof of [3, Theorem 3].

Lemma 1. *Let R be a Noetherian ring and $\{E_r^{pq}\}$ a first quadrant cohomology spectral sequence (starting with E_a, for some $a \geq 1$) converging to H^* in the category of R-modules. For a fixed integer n, suppose H^n is finitely generated and $E_a^{p,q}$ is finitely generated for all $p < n$ and $q \geq 0$. Then $E_a^{n,0}$ is finitely generated.*

Proof. If $n = 0$ then $E_a^{00} = H^0$ is finitely generated. Suppose $n > 0$. First note that E_r^{pq} is finitely generated for any $p < n$, $q \geq 0$, and $r \geq a$, since E_r^{pq} is a subquotient of E_a^{pq}. Also, as $E_\infty^{n,0}$ is a isomorphic to a submodule of H^n, $E_\infty^{n,0}$ is finitely generated. Now since $\{E_r^{pq}\}$ is a first quadrant spectral sequence (in particular, since there are no nonzero terms below the p-axis), there is an exact sequence

$$E_r^{n-r,r-1} \to E_r^{n,0} \to E_{r+1}^{n,0} \to 0$$

for all $r \geq a$. As $E_r^{n,0} = E_\infty^{n,0}$ for sufficiently large r (and thus is finitely generated), we can work backwards to see that $E_r^{n,0}$ is finitely generated for all $r \geq a$. □

We now prove the change of ring principle for cofiniteness.

Proposition 2. *Let R be a Noetherian ring and S a module finite R-algebra. Let I be an ideal of R and M an S-module. Then M is I-cofinite (as an R-module) if and only if M is IS-cofinite (as an S-module).*

Proof. First note that $\operatorname{Supp}_R M \subseteq V(I)$ if and only if $\operatorname{Supp}_S M \subseteq V(IS)$. Now consider the Grothendieck spectral sequence (see [11, Theorem 11.65], for example)

$$E_2^{pq} = \operatorname{Ext}_S^p(\operatorname{Tor}_q^R(S,R/I),M) \Rightarrow \operatorname{Ext}_R^{p+q}(R/I,M).$$

Suppose first that M is IS-cofinite. Then $E_2^{p,0} = \operatorname{Ext}_S^p(S/IS,M)$ is finitely generated for all p. Since $\operatorname{Supp}\operatorname{Tor}_q^R(S,R/I) \subseteq \operatorname{Supp} S/IS$ for all q, E_2^{pq} is finitely generated for all p and q by Proposition 1. Since the spectral sequence is bounded, it follows that $\operatorname{Ext}_R^n(R/I,M)$ is finitely generated for all n.

Conversely, suppose that M is I-cofinite. We use induction on n to show $E_2^{n,0} = \operatorname{Ext}_S^n(S/IS,M)$ is finitely generated. Now $E_2^{00} = \operatorname{Hom}_S(S/IS,M) \cong \operatorname{Hom}_R(R/I,M)$ is finitely generated. Suppose that $n > 0$ and $E_2^{p,0}$ is finitely generated for all $p < n$. By Proposition 1, E_2^{pq} is finitely generated for all $p < n$ and $q \geq 0$. Since $H^n = \operatorname{Ext}_R^n(R/I,M)$ is finitely generated, $E_2^{n,0}$ is finitely generated by Lemma 1. □

As a final preparation for the proof of Theorem 1, we need the following fact.

Lemma 2. *Let (R,m) be a local ring and S the m-adic completion of R. Let I be an ideal of R and M an R-module. Then $H_I^i(M)$ is I-cofinite if and only if $H_{IS}^i(M \otimes_R S)$ is IS-cofinite.*

Proof. Since $\operatorname{Ext}_R^i(R/I,H_I^i(M)) \otimes_R S \cong \operatorname{Ext}_S^i(S/IS,H_{IS}^i(M \otimes_R S))$, it is enough to see that an R-module N is finitely generated if and only if $N \otimes_R S$ is finitely generated as an S-modules. If N is finitely generated, the implication is obvious. If $N \otimes_R S$ is finitely generated then, using the faithful flatness of S, one can see that any ascending chain of submodules of N must stabilize. □

Theorem 1 now follows readily.

Proof of Theorem 1. By Lemma 2 we may assume R is complete. Thus, R is the homomorphic image of a regular local ring T. Let J be a dimension one ideal of T such that $JR = I$. Then $H_J^j(M)$ is J-cofinite by [7, Theorem 4.1] for all j. By Proposition 2, $H_I^j(M) \cong H_J^j(M)$ is I-cofinite for all j. □

If N is an R-module then the ith Bass number of N with respect to p is defined to be $\mu_i(p,N) = \dim_{k(p)} \operatorname{Ext}_{R_p}^i(k(p),N_p)$, where $k(p) = (R/p)_p$. If M is finitely generated and I is a zero-dimensional ideal then the Bass numbers of $H_I^i(M)$ are finite since $H_I^i(M)$ is Artinian. However, as Hartshorne's example shows, this does not hold for arbitrary ideals and modules, even over a complete regular local ring. In the special case that $M = R$, Huneke and Sharp proved that if R is a regular local ring of characteristic p and I is an ideal of R, then the Bass numbers of $H_I^i(R)$ are finite for all i [8, Theorem 2.1]. Lyubeznik proved this same result in the case R is a regular local ring containing a field of characteristic 0 [9, Corollary 3.6]. In [1], it is proved that if R is a complete local Gorenstein domain, I is a dimension one ideal and M is a Matlis reflexive R-module (i.e., $\mathrm{Hom}_R(\mathrm{Hom}_R(M,E),E) \cong M$ where $E = E_R(R/m)$), then the Bass numbers of $H_I^j(M)$ are finite. Using Theorem 1, we can prove the following.

Corollary 2. *Let R be a Noetherian ring, I a dimension one ideal of R, and M a finitely generated R-module. Then $\mu_i(p,H_I^j(M)) < \infty$ for all integers i,j and $p \in \mathrm{Spec}(R)$.*

Proof. If $p \not\supseteq I$, then $\mu_i(p,H_I^j(M))=0$. If $p \supseteq I$ we can localize and assume $p=m$. By Theorem 1, $\mathrm{Ext}_R^i(R/I,H_I^j(M))$ is finitely generated for all i,j. Thus, $\mathrm{Ext}_R^i(R/m, H_I^j(M))$ is finitely generated for all i,j by Corollary 1. □

Another question Hartshorne addressed in [6] was the following: if R is a complete regular local ring and P is a prime ideal, do the P-cofinite modules form an abelian subcategory of the category of all R-modules? That is, if $f : A \to B$ is an R-module map of P-cofinite modules, are $\ker f$ and $\operatorname{coker} f$ P-cofinite? Hartshorne gave the following counterexample: let $R = k[[x,y,u,v]]$, $P = (x,u)$ and $M = R/(xy-uv)$. Applying the functor $H_P^0(-)$ to the exact sequence

$$0 \to R \xrightarrow{xy-uv} R \to M \to 0,$$

we get the exact sequence

$$\cdots \to H_P^2(R) \xrightarrow{f} H_P^2(R) \to H_P^2(M) \to 0.$$

Since $H_P^j(R) = 0$ for all $j \neq 2$, one can show (using a collapsing spectral sequence) that $\mathrm{Ext}_R^i(R/P,H_P^2(R)) \cong \mathrm{Ext}_R^{i+2}(R/P,R)$ for all i. Thus, $H_P^2(R)$ is P-cofinite. However, as mentioned previously, $\operatorname{coker} f = H_P^2(M)$ is not P-cofinite. On the positive side, Hartshorne proved that if P is a dimension one prime ideal of a complete regular local ring then the answer to his question is yes. Using Proposition 2, we can extend this result to arbitrary complete local rings.

Theorem 2. *Let R be a complete local ring and P a dimension one prime ideal of R. Then the P-cofinite modules form an abelian subcategory of the category of all R-modules.*

Proof. Let $f : M \to N$ be a map of P-cofinite modules. Since R is complete there exists a regular local ring T and a dimension one prime ideal Q of T such that R is a quotient of T and $QT = P$. Since M and N are Q-cofinite T-modules, $\ker f$ and $\operatorname{coker} f$ are Q-cofinite by Hartshorne's theorem [6, Proposition 7.6]. Therefore, $\ker f$ and $\operatorname{coker} f$ are P-cofinite by Proposition 2. □

We now turn our attention to proving Theorem 3. The techniques are essentially those of Sharp [12] and Yassemi [15]. Let (R,m) be a local ring, M an R-module, and $E = E_R(R/m)$ the injective hull of R/m. Following [15], we define a prime p to be a *coassociated prime* of M if p is an associated prime of $M^{\vee} = \mathrm{Hom}_R(M,E)$. We denote the set of coassociated primes of M by $\mathrm{Coass}_R M$ (or simply $\mathrm{Coass}\, M$ if there is no ambiguity about the underlying ring). Note that $\mathrm{Coass}\, M = \emptyset$ if and only if $M = 0$. We first make a couple of preliminary remarks.

Remark (Vasconcelos [15, Theorem 1.22]). Let (R,m) be a Noetherian local ring, M a finitely generated R-module and N an arbitrary R-module. Then $\mathrm{Coass}(M \otimes_R N) = \mathrm{Supp}\, M \cap \mathrm{Coass}\, N$.

Proof. Note that $(M \otimes_R N)^{\vee} \cong \mathrm{Hom}_R(M, N^{\vee})$. Therefore,

$$\begin{aligned}\mathrm{Coass}(M \otimes_R N) &= \mathrm{Ass}(\mathrm{Hom}_R(M,N^{\vee}))\\ &= \mathrm{Supp}\, M \cap \mathrm{Ass}\, N^{\vee} \qquad \text{(e.g., [2, IV.1.4, Proposition 10])}\\ &= \mathrm{Supp}\, M \cap \mathrm{Coass}\, N. \qquad \square\end{aligned}$$

Remark 2. Let R be a local ring of dimension d, I an ideal of R and M an R-module. Then $H_I^d(M) \cong M \otimes_R H_I^d(R)$.

Proof. Since $H_I^d(-)$ is a right exact functor, this remark is an immediate consequence of Watts' Theorem [11, Theorem 3.33]. Here is a more direct proof: since R is local, there exist elements $\underline{x} = x_1, \ldots, x_d \in I$ which generate I up to radical. Then $H_I^i(M) = H_{(\underline{x})}^i(M)$ for all i. Using the Čech complex to compute $H_{(\underline{x})}^d(R)$, we see there is an exact sequence

$$\bigoplus_i R_{x_1 \cdots \hat{x}_i \cdots x_d} \to R_{x_1 \cdots x_d} \to H_{(\underline{x})}^d(R) \to 0.$$

Tensoring this sequence with M, we get the exact sequence

$$\bigoplus_i M_{x_1 \cdots \hat{x}_i \cdots x_d} \xrightarrow{f} M_{x_1 \cdots x_d} \to M \otimes_R H_{(\underline{x})}^d(R) \to 0.$$

Since $\mathrm{coker}\, f = H_{(\underline{x})}^d(M)$, we see that $H_I^d(M) \cong M \otimes_R H_I^d(R)$. $\square$

The next result is essentially a module version of [12, Theorem 3.4] combined with [15, Theorem 1.16]. As in [12], we make repeated use of the Hartshorne–Lichtenbaum vanishing theorem (HLVT): if (R,m) is a complete local ring of dimension d and I is an ideal of R, then $H_I^d(R) \neq 0$ if and only if $I + p$ is m-primary for some prime ideal p such that $\dim R/p = d$ [5, Theorem 3.1].

Lemma 3. *Let (R,m) be a complete Noetherian local ring, I an ideal of R and M a finitely generated R-module of dimension n. Then*

$$\mathrm{Coass}\, H_I^n(M) = \{p \in \mathrm{V}(\mathrm{Ann}_R M) \mid \dim R/p = n \text{ and } \sqrt{I + p} = m\}.$$

Proof. Let $S = R/\operatorname{Ann}_R M$ and $E_S = \operatorname{Hom}_R(S,E)$ the injective hull of the residue field of S. Observe that

$$\begin{aligned}\operatorname{Hom}_S(H^n_{IS}(M),E_S) &\cong \operatorname{Hom}_R(H^n_I(M),E_S)\\ &\cong \operatorname{Hom}_R(H^n_I(M)\otimes_R S,E)\\ &\cong \operatorname{Hom}_R(H^n_I(M),E).\end{aligned}$$

Consequently, $\operatorname{Coass}_R H^n_I(M) = \pi(\operatorname{Coass}_S H^n_{IS}(M))$ where $\pi : \operatorname{Spec} S \to \operatorname{Spec} R$. Thus, we may assume that $\operatorname{Ann}_R M = 0$ and $n = \dim R$. By Remarks 1 and 2, we have $\operatorname{Coass} H^n_I(M) = \operatorname{Coass}(M \otimes_R H^n_I(R)) = \operatorname{Coass} H^n_I(R)$, so it is enough to prove the result in the case $M = R$. By HLVT, both sets are empty if $H^n_I(R) = 0$, so assume that $H^n_I(R) \neq 0$. Let $q \in \operatorname{Coass} H^n_I(R)$. By the remark, $q \in \operatorname{Coass}(R/q \otimes_R H^n_I(R))$. In particular, $R/q \otimes_R H^n_I(R) \cong H^n_I(R/q) \neq 0$. Thus, $n = \dim R/q$ and $I + q$ is m-primary (by HLVT). Now suppose $\dim R/q = n$ and $\sqrt{I+q} = m$. By reversing the above argument we get that $R/q \otimes_R H^n_I(R) \neq 0$. Let $p \in \operatorname{Coass}(R/q \otimes_R H^n_I(R))$. By Remark 1, $p \supseteq q$ and $p \in \operatorname{Coass} H^n_I(R)$. We have already shown that every coassociated prime of $H^n_I(R)$ is a minimal prime of R. Hence $p = q$ and $q \in \operatorname{Coass} H^n_I(R)$, which completes the proof. □

We now show that $H^{\dim M}_I(M)$ is I-cofinite:

Theorem 3. *Let (R,m) be Noetherian local ring, I an ideal of R and M a finitely generated R-module of dimension n. Then $H^n_I(M)$ is I-cofinite. In fact,* $\operatorname{Ext}^i_R(R/I,H^n_I(M))$ *has finite length for all i.*

Proof. By Lemma 2 we may assume that R is complete. Let $\operatorname{Coass} H^n_I(M) = \{p_1,\ldots, p_k\}$. Since $H^n_I(M)$ is Artinian (see [12, Theorem 3.3]), $H^n_I(M)^\vee$ is finitely generated. Hence, $\operatorname{Supp} H^n_I(M)^\vee = V(p_1 \cap \cdots \cap p_k)$. By Matlis duality, $\operatorname{Ext}^i_R(R/I,H^n_I(M))$ has finite length if and only if $\operatorname{Ext}^i_R(R/I,H^n_I(M))^\vee \cong \operatorname{Tor}^R_i(R/I,H^n_I(M)^\vee)$ [11, Theorem 11.57] has finite length. Since $\operatorname{Tor}^R_i(R/I,H^n_I(M)^\vee)$ is finitely generated, it is enough to show its support is contained in $\{m\}$. But

$$\begin{aligned}\operatorname{Supp}\operatorname{Tor}^R_i(R/I,H^n_I(M)^\vee) &\subseteq V(I)\cap \operatorname{Supp} H^n_I(M)^\vee\\ &= V(I)\cap V(p_1\cap\cdots\cap p_k)\\ &= V(I+(p_1\cap\cdots\cap p_k))\\ &= \{m\} \qquad \text{(by Lemma 3).}\end{aligned}$$ □

References

[1] R. Belshoff, S. Slattery and C. Wickham, The local cohomology modules of Matlis reflexive modules are almost cofinite, Proc. Amer. Math. Soc., to appear.

[2] N. Bourbaki, Algèbre Commutative (Hermann, Paris, 1961).

[3] D. Delfino, On the cofiniteness of local cohomology modules, Math. Proc. Camb. Phil. Soc. 115 (1994) 79–84.

[4] D. Delfino, Cofiniteness and vanishing of local cohomology modules, and colength of conductor ideals, Ph.D. Dissertation, Purdue University, 1994.

[5] R. Hartshorne, Cohomological dimension of algebraic varieties, Ann. Math. 88 (1968) 403–450.
[6] R. Hartshorne, Affine duality and cofiniteness, Inv. Math. 9 (1970) 145–164.
[7] C. Huneke and J. Koh, Cofiniteness and vanishing of local cohomology modules, Math. Proc. Cambridge Phil. Soc. 110 (1991) 421–429.
[8] C. Huneke and R. Sharp, Bass numbers of local cohomology modules, Trans. Amer. Math. Soc. 339 (1993) 765–779.
[9] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra), Inv. Math. 113, 41–55.
[10] L. Melkersson, Some applications of a criterion for artinianness of a module, J. Pure Appl. Algebra 101 (1995) 291–303.
[11] J. Rotman, An Introduction to Homological Algebra (Academic Press, Orlando, FL, 1979).
[12] R. Sharp, On the attached prime ideals of certain Artinian local cohomology modules, Proc. Edinburgh Math. Soc. 24 (1981).
[13] W. Vasconcelos, Divisor Theory in Module Categories (North-Holland, Amsterdam, 1974).
[14] C. Weibel, An Introduction to Homological Algebra (Cambridge Univ. Press, Cambridge, 1994).
[15] S. Yassemi, Coassociated primes, preprint.